Math 250 3.9 Related Rates

Objectives

1) Use chain rule to find rates of change with respect to time
   a. Assume all variables change as time changes, unless stated otherwise.
   b. Take derivatives $\frac{d}{dt}$ of all variables
2) Find related rates equations
   a. Given an equation that describes the relationship among several variables, the **related rates equation** is the derivative of the original equation with respect to time t, and it shows how the rates of change are related to each other.
   b. Assume all variables change with respect to time, unless stated otherwise.
   c. Take derivatives $\frac{d}{dt}$ of both sides of the main equation.
3) Use related rates equations to solve problems
   a. Draw a diagram or sketch
   b. List all variables
   c. Write one or more formulas or equations which describe the situation and are true at *all* time t
   d. If one equation or relationship can be substituted to eliminate a variable, do this
   e. Recognize which variables are changing
   f. Find the related rates equation by taking derivatives $\frac{d}{dt}$
   g. Substitute given values of variable at the instant of interest
   h. Substitute given rates of change for derivatives at the instant of interest
   i. Solve algebraically for one unknown
   j. Use units to check work and answer question
   k. Check that the answer is reasonable—especially signs.
      i. If a rate of change is negative, that quantity is decreasing.
      ii. If a rate of change is positive, that quantity is increasing.

Examples and Practice

1) Find derivatives
   a. $\frac{d}{dx}\left(5x^3\right)$
   b. $\frac{d}{dt}\left(5x^3\right)$
   c. $\frac{d}{dt}\left(5x^3+y^2\right)$
   d. If $xy=4$, find $\frac{dx}{dt}$ when $x=8$ and $\frac{dy}{dt}=-2$

2) Suppose a point is moving along the graph of $y=\sqrt{x}$ in such a way that $\frac{dx}{dt}=-1$ cm/sec. Find the rate at which the y-coordinate is changing at the moment when $x=9$ cm.

3) Bacteria are growing in a circular colony one bacterium thick. The bacteria are growing at a constant rate, thus making the area of the colony increase at a constant rate of 12 mm^2/hr. Determine how fast the radius of the circle is changing at the moment when it equals 3 mm.

4) Johnny B. Good blows up a spherical balloon. In order for the radius to increase at 2 $cm/\sec$, how fast must Johnny blow air into the balloon when $r = 3$ cm?

5) A point is moving along the graph of $y = \sqrt{x}$ in such a awy that $\frac{dx}{dt} = 0.5$ cm/sec. Find the rate at which the distance between the point and the origin is changing when $x = 4$.

6) The edges of a cube are expanding at a rate of 1.5 cm/min. Find the rate at which the a) volume and b) surface area are changing when the edge is 2 cm long.

7) A reservoir is in the shape of a cone, vertex down. The radius at the top is 20 meters, and the height of the reservoir is 15 meters. Water is pouring out of the reservoir at a rate of 2 $m^3/\min$. Find the rate at which the height of the water is changing at the moment when the height is 10 meters.

8) A 20-foot ladder leans against a wall. The base of the ladder is being pushed toward the wall at a rate of 0.2 ft/sec. Find:
   a. The rate at which the top of the ladder is moving up the wall at the moment when the base of the ladder is 6 feet from the wall.
   b. The rate at which the area of the triangle formed by the wall, ladder, and ground is changing at the same moment as in part a).

9) A man is walking away from a lamppost at 5 ft/sec. The lamp is 15 feet tall. The man is 6 feet tall.
   a. When he is 10 feet away from the base of the light, at what rate is the tip of his shadow moving?
   b. When he is 10 feet away from the base of the light, at what rate is the length of his shadow changing?

Geometry Formulas

DEFINITIONS

The **perimeter** is the sum of the lengths of all the sides of a figure.

The **area** is the amount of space enclosed by a two-dimensional figure measured in units squared.

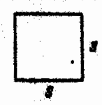
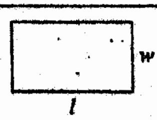
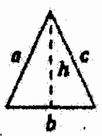
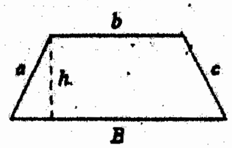
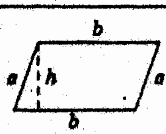
The **surface area** of a solid is the sum of the areas of the surfaces of a three-dimensional figure.

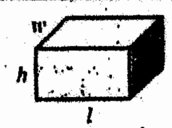
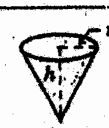
The **volume** is the amount of space occupied by a three-dimensional figure measured in units cubed.

The **radius** r of a circle is the line segment that extends from the center of the circle to any point on the circle.

The **diameter** of a circle is any line segment that extends from one point on the circle through the center to a second point on the circle. The diameter is two times the length of the radius, $d = 2r$.

In circles, we use the term **circumference** to mean the perimeter.

Plane Figures		Formulas
Square		Area: $A = s^2$ Perimeter: $P = 4s$
Rectangle		Area: $A = lw$ Perimeter: $P = 2l + 2w$
Triangle		Area: $A = \frac{1}{2}bh$ Perimeter: $P = a + b + c$
Trapezoid		Area: $A = \frac{1}{2}h(B + b)$ Perimeter: $P = a + b + c + B$
Parallelogram		Area: $A = bh$ Perimeter: $P = 2a + 2b$
Circle		Area: $A = \pi r^2$ Circumference: $C = 2\pi r = \pi d$

Solids		Formulas
Cube		Volume: $V = s^3$ Surface Area: $S = 6s^2$
Rectangular Solid		Volume: $V = lwh$ Surface Area: $S = 2lw + 2lh + 2wh$
Sphere		Volume: $V = \frac{4}{3}\pi r^3$ Surface Area: $S = 4\pi r^2$
Right Circular Cylinder		Volume: $V = \pi r^2 h$ Surface Area: $S = 2\pi r^2 + 2\pi rh$
Cone		Volume: $V = \frac{1}{3}\pi r^2 h$ Surface Area: $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$

Units Analysis

Objective: Use units to check answers and formulas

Units describe what type of items or measurements are used. Ex: feet, minutes, liters, revolutions, miles per hour, dollars per square foot, cubic feet per minute.

Rules for units or dimensional analysis

1. To add or subtract units, the terms being combined must have the same units, and the result has those same units. (This is the same as "combining like terms".)
   Example: feet + feet = feet
   Example: 2 apples + 3 apples = 5 apples

2. When multiplying units, the dimension of the units increases. The exponent on the units is called the "dimension".
   Example: feet x feet = square feet = $feet^2$

3. The word "per" in the units means divide.
   Example: miles per hour = $\frac{miles}{hour}$ (the rate a car moves)
   Example: dollars per square foot = $\frac{dollars}{feet^2}$ (the cost of carpeting)
   Example: cubic feet per minute = $\frac{feet^3}{\text{min}}$ (rate a bathtub fills with water)

4. If you multiply a "per" expression by another expression with the same units as the denominator, the units cancel. (This is like canceling or dividing out a common factor.)
   Example: $\frac{miles}{hour} \cdot hours = miles$ (because 'hour' and 'hours' are the same, they cancel out)
   Example: $\frac{\$2}{foot^2} \cdot 17\,feet^2 = \$34$ (because square feet cancel out)

5. To check an equation or formula
   a. The simplified units must be the same on the left side as on the right side.
   b. Substitute the units for each quantity into the equation.
   c. Constants and constant coefficients have no units.

   Example: $D = R \cdot T$, where distance D is given in miles, rate R is given in miles per hour, and time is given in hours, means the dimensions in the equation are: $miles = \frac{miles}{hour} \cdot hours$, the hours cancel, and we have miles = miles.

Example: Suppose we mis-remembered the formula as $D \cdot R = T$. Dimensional analysis gives $miles \cdot \frac{miles}{hour} = hours$, but the left side simplifies to $\frac{miles^2}{hour} \neq hours$, helping us realize that we have the formula wrong, because the units on the left are not the same as the units on the right.

6. Exponents or dimensions tell us the shape of the concept being calculated
   a. A one-dimensional calculation is length of a line or a curved line, measured in one-dimensional units, like feet or meters.

      Example: the distance from point A to point B

      Example: the length of one side of a rectangle

      Example: the distance around the edge of a circle, the circumference $C = 2\pi r$, where C and r are measured in meters and 2 and π are constant coefficients having no units, has dimensional analysis $meters = (no\,units)(no\,units) \cdot meters$

      Example: the cost of fencing $\frac{\$2}{meter}$ tells us we want length or perimeter.

   b. A two-dimensional calculation is area, the flat space contained in a flat object, measured in two-dimensional units, like square feet= $feet^2$

      Example: the area of a rectangle $A = L \cdot W$ has dimensional analysis $feet^2 = feet \cdot feet$

      Example: the cost per square foot of carpet $\frac{\$2}{foot^2}$ tells us that we need the area of the space to be carpeted

   c. A three-dimensional calculation is volume, the voluminous space contained within a real-world object, measured in three-dimensional units, like cubic meters = $meters^3$

      Example: the volume of a rectangular solid $V = L \cdot W \cdot H$ has dimensional analysis $feet^3 = feet \cdot feet \cdot feet$

Practice:
Use dimensional analysis to determine if the following formulas and calculations might be correct.

1) The perimeter of a rectangle is $P = 2L + 2W$
2) The surface area of a rectangular solid is $A = 2LW + 2WH + 2LH$
3) The volume of a sphere is $V = \frac{4}{3}\pi r^3$
4) The total cost of fencing = cost per foot times the number of square feet.

Math 250 Related Rates

Review. Find each of the following:

1 a. $\frac{d}{dx}(5x^3)$ $= \boxed{15x^2}$

b. $\frac{d}{dt}(5x^3)$ $= \boxed{15x^2 \frac{dx}{dt}}$

c. $\frac{d}{dt}(5x^3 + y^2)$ $= \boxed{15x^2 \frac{dx}{dt} + 2y\frac{dy}{dt}}$

*Both (b) and (c) require CHAIN RULE because the derivative is with respect to t, yet the expressions are not in terms of t.

d. If $xy = 4$, find $\frac{dx}{dt}$ when $x = 8$ and $\frac{dy}{dt} = -2$.

To find y, plug in.
$8y = 4 \Rightarrow y = \frac{1}{2}$

$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$

$8(-2) + \frac{1}{2}\cdot\frac{dx}{dt} = 0$

$\frac{1}{2}\frac{dx}{dt} = 16$

$\boxed{\frac{dx}{dt} = 32}$

RELATED RATES problems concern rates of change of two or more related variables that are changing with respect to *time*.

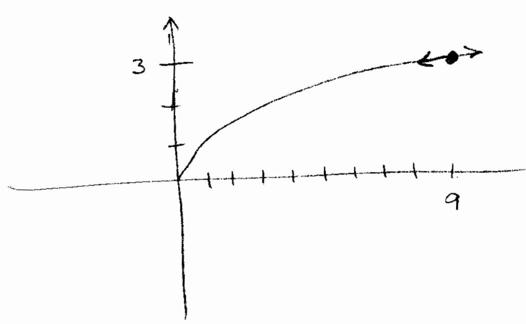
2 For example: Suppose a point is moving along the graph of $y = \sqrt{x}$ in such a way that $\frac{dx}{dt} = -1\ cm/\sec$. Find the rate at which the y coordinate is changing (i.e. find dy/dt) at the moment when $x = 9$ cm.

$\frac{dy}{dt} = \frac{1}{2}x^{-1/2}\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{1}{2}(9)^{-1/2}\cdot(-1)$

$= \frac{-1}{2\cdot 3}$

$\boxed{\frac{dy}{dt} = \frac{-1}{6}\ cm/sec}$



$\frac{dx}{dt} < 0 \Rightarrow$ particle moving left

$\frac{dy}{dt} < 0 \Rightarrow$ particle moving down

To solve Related Rates Problems, first list the following:

a. What you are GIVEN. This means the specific information listed in the problem.

b. What you are to FIND.

c. What you KNOW. This means any formulas (such as area, volume, Pythagorean Theorem, etc.) that would be appropriate for the situation. A picture might also help.

THEN, check to see if the formula you wrote in the KNOW part relates the variables that you have information about in the GIVEN, and what you want to FIND. If so, differentiate the formula from the KNOW *with respect to time!*

3 Bacteria are growing in a circular colony one bacterium thick. The bacteria are growing at a constant rate, thus making the area of the colony increase at a constant rate of 12 mm^2 / hr. Determine how fast the radius of the circle is changing at the moment when it equals 3 *mm*.

GIVEN: $\frac{dA}{dt} = 12\, mm^2/hr$, $r = 3mm$

FIND: $\frac{dr}{dt}$

KNOW: area of circle $A = \pi r^2$

Solve:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$12 = 2\pi \cdot 3 \cdot \frac{dr}{dt}$$

$$\frac{12}{6\pi} = \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{2}{\pi}\, mm/hr}$$

4 Johnny B. Good blows up a spherical balloon. In order for the radius to increase at 2 cm/sec, how fast must Johnny blow air into the balloon when r = 3 cm?

GIVEN: $\frac{dr}{dt} = 2\, cm/sec$, $r = 3\,cm$

FIND: $\frac{dV}{dt}$

KNOW: volume of sphere $V = \frac{4}{3}\pi r^3$

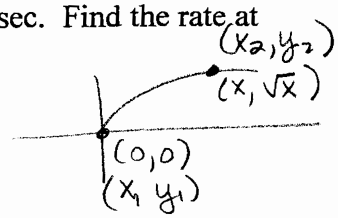
Solve:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi (3)^2 (2)$$

$$\boxed{\frac{dV}{dt} = 72\pi\, cm^3/sec}$$

(5) 3. A point is moving along the graph of $y = \sqrt{x}$ in such a way that $\frac{dx}{dt} = 0.5$ cm/sec. Find the rate at which the *distance* between the point and the origin is changing when $x = 4$.



GIVEN: $\frac{dx}{dt} = 0.5$ cm|sec, $x=4$, $y=\sqrt{x}$

FIND: $\frac{dD}{dt}$

KNOW: distance formula $D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

Solve:

$$D = \sqrt{(x-0)^2 + (\sqrt{x}-0)^2}$$

$$D = \sqrt{x^2+x} = (x^2+x)^{1/2}$$

$$\frac{dD}{dt} = \tfrac{1}{2}(x^2+x)^{-1/2}(2x+1)\frac{dx}{dt}$$

$$\frac{dD}{dt} = \tfrac{1}{2}(4^2+4)^{-1/2}(2\cdot 4+1)\cdot(0.5) = \frac{9\cdot 1}{2\cdot 2\sqrt{20}} = \boxed{\frac{9}{8\sqrt{5}}\ \text{cm/sec}}$$

(6) The edges of a cube are expanding at a rate of 1.5 cm/min. Find the rate at which the (a) volume, and (b) surface area are changing WHEN the edge is 2 cm long.

a. For Volume:

GIVEN: $\frac{dx}{dt} = 1.5$ cm/m, $x = 2$cm

FIND: $\frac{dV}{dt}$

KNOW: $V = x^3$

Solve:

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$\frac{dV}{dt} = 3(2)^2(1.5) = \boxed{18\ \text{cm}^3/\text{min}}$$

b. For the Surface Area:

GIVEN: $\frac{dx}{dt} = 1.5$ cm/min, $x = 2$ cm

FIND: $\frac{dA}{dt}$

KNOW: $A = 6x^2$

Solve:

$$\frac{dA}{dt} = 12x\frac{dx}{dt}$$

$$\frac{dA}{dt} = 12(2)(1.5) = \boxed{36\ \text{cm}^2/\text{min}}$$

Math 250

⑤ Another (longer?) approach: Differentiate pieces and subst

Given $\frac{dx}{dt} = 0.5\ \text{cm/sec}$

$x = 4$

$y = \sqrt{x}$

Find $\frac{dD}{dt}$

Know $D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ distance formula.

substitute $(x_1, y_1) = (0,0)$

$(x_2, y_2) = (x, y)$ any point on the curve.

$$D = \sqrt{(x-0)^2 + (y-0)^2}$$

$$D = \sqrt{x^2+y^2}$$

If we differentiate wrt t now ....

$$\underset{\substack{\uparrow \\ \text{find} \checkmark}}{\frac{dD}{dt}} = \frac{1}{2}(\underset{\substack{\uparrow \\ \text{Know} \checkmark}}{x^2}+\underset{\substack{\uparrow \\ \text{dont Know}}}{y^2})^{-1/2}\left(2x\underset{\substack{\uparrow \\ \text{Know} \checkmark}}{\frac{dx}{dt}} + 2y\underset{\substack{\Uparrow \\ \text{Trouble.}}}{\frac{dy}{dt}}\right)$$

we need $\frac{dy}{dt}$! and y...

Since $y = \sqrt{x} = x^{1/2}$

Differentiate: $\frac{dy}{dt} = \frac{1}{2}x^{-1/2}\cdot\frac{dx}{dt}$

subst for $\frac{dy}{dt}$

$$\frac{dD}{dt} = \frac{1}{2}(x^2+y^2)^{-1/2}\left(2x\cdot\frac{dx}{dt} + 2y\cdot\frac{1}{2}x^{-1/2}\frac{dx}{dt}\right)$$

Subst $x=4$, $y=\sqrt{4}$, $\frac{dx}{dt} = 0.5$

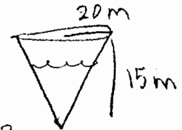
$$\frac{dD}{dt} = \frac{1}{2}\left(4^2+2^2\right)^{-1/2}\left(2\cdot4\cdot(0.5) + 2\cdot2\cdot\frac{1}{2}\cdot(4)^{-1/2}\cdot0.5\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{20}}(4+0.5) = \frac{1}{2}\cdot\frac{1}{2\sqrt{5}}\cdot\frac{9}{2} = \frac{9}{8\sqrt{5}}\ \text{cm/sec}$$

Math 250 Related Rates

7. A reservoir is in the shape of a cone, vertex down. The radius at the top is 20 meters, and the height of the reservoir is 15 meters. Water is pouring OUT of the reservoir at a rate of 2 m^3/min. Find the rate at which the HEIGHT of the water is changing at the moment when the height is 10 meters.

DRAW A PICTURE:



$\frac{dV}{dt} = -2\frac{m^3}{min}$

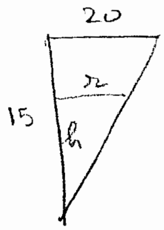
Given: $h = 10m$, $\frac{dV}{dt} = -2\ m^3/min$

Find. $\frac{dh}{dt}$

Know: $R = 20$ $H = 15$ reservoir

similar triangles w/ r, h of water

$V = \frac{1}{3}\pi r^2 h$



$\frac{20}{15} = \frac{r}{h}$.

$r = \frac{4}{3}h$.

$V = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h$

$V = \frac{1}{3}\pi \cdot \frac{16}{9}h^3$

$V = \frac{16\pi}{27}h^3$

Remove: r

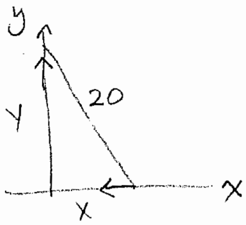
$\frac{dV}{dt} = \frac{16\pi}{9}h^2\frac{dh}{dt}$

$-2 = \frac{16\pi}{9}(10^2)\frac{dh}{dt}$

$\frac{-18}{1600\pi} = \frac{dh}{dt} = \boxed{\frac{-9}{800\pi}\ m/min}$

8. A 20 ft. ladder leans against the wall. The base of the ladder is being pushed TOWARD the wall at a rate of 0.2 ft/sec. Find:

a. The rate at which the top of the ladder is moving up the wall at the moment when the base of the ladder is 6 ft. from the wall.



$\frac{dx}{dt} = -0.2\ ft/sec$, $x = 6$

find $\frac{dy}{dt} > 0$

$x^2 + y^2 = 20$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(6)\cdot(-0.2) + 2(y)\frac{dy}{dt} = 0$

$6^2 + y^2 = 20^2$

$y = \sqrt{364} = 2\sqrt{91}$

$-2.4 + 2\cdot 2\sqrt{91}\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{(2.4)}{4\sqrt{91}} = \boxed{\frac{3}{5\sqrt{91}}\ ft/sec}$

b. The rate at which the AREA of the triangle formed by the wall, ladder, and ground is changing at the same moment as in part (a).

$\frac{dx}{dt} = -0.2\ ft/sec$, $x = 6$, $y = 2\sqrt{91}$

$A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right]$

$\frac{dA}{dt} = \frac{1}{2}\left[6\cdot\left(\frac{3}{5\sqrt{91}}\right) + 2\sqrt{91}(-0.2)\right]$

$\boxed{\frac{dA}{dt} = \frac{9}{5\sqrt{91}} - \frac{\sqrt{91}}{5}\ ft^2/sec}$ $= 4$ $\frac{9\sqrt{91}}{455} - \frac{\sqrt{91}}{5} = \left(\frac{9}{455} - \frac{1}{5}\right)\sqrt{91}$

$= \boxed{\frac{-82\sqrt{91}}{455}\ ft^2/sec}$

7 Alternate approach: differentiate pieces + substitute

Given: $h = 10$ m

$\frac{dV}{dt} = -2 \, m^3/\text{min}$

Find: $\frac{dh}{dt}$

Know: $R = 20$, $H = 15$ on reservoirs
Similar triangles with r, h of water
$V = \frac{1}{3}\pi r^2 h$

Differentiate $V = \frac{1}{3}\pi r^2 h$ — requires product rule!

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}\cdot h + r^2 \cdot 1 \cdot \frac{dh}{dt}\right)$$

given ✓

don't know r

don't know $\frac{dr}{dt}$

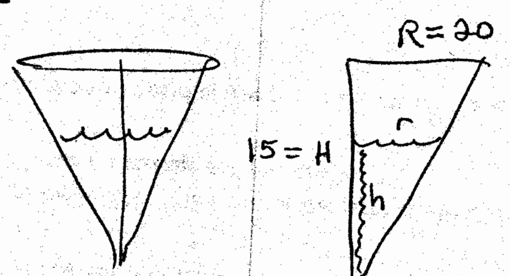
given ✓

find ✓

Trouble.

Similar Triangles

$$\frac{20}{15} = \frac{r}{h}$$



$$\frac{4}{3}h = r$$

Differentiate! $\frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt}$

subst $r = \frac{4}{3}(10) = \frac{40}{3}$

Then substitute back into $\frac{dV}{dt}$ equation

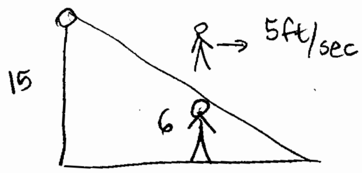
$$-2 = \frac{1}{3}\pi\left(2\cdot\frac{40}{3}\cdot\frac{4}{3}\cdot\frac{dh}{dt}\cdot 10 + \left(\frac{40}{3}\right)^2\cdot\frac{dh}{dt}\right)$$

solve for $\frac{dh}{dt}$

$$\frac{-6}{\pi} = \left(\frac{3200}{9}\cdot\frac{dh}{dt} + \frac{1600}{9}\cdot\frac{dh}{dt}\right)$$

$$\frac{-6}{\pi} = \frac{4800}{9}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-6\cdot 9}{\pi\cdot 4800} = \boxed{\frac{-9}{800\pi} \text{ m/min}}$$

(9)

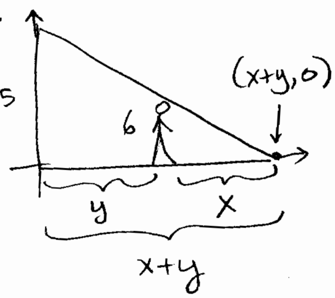


A man is walking away from a lamp post at 5ft/sec.
The lamp is 15 ft tall.
The man is 6 ft tall.

a) When he is 10 ft away from the base of the light, at what rate is the tip of his shadow moving?

Here are
Four ways to draw diagram → notice that the rate of change of <u>tip of shadow</u> is the rate of change of the <u>coordinate</u>.

I



$$\frac{15}{6} = \frac{x+y}{x}$$

$$\frac{dy}{dt} = 5$$

find $\frac{d}{dt}(x+y)$

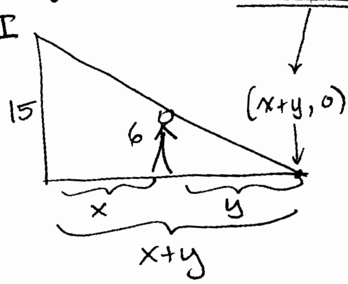
$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = \frac{dx}{dt} + 5$$

cross-mult:

$$15x = 6x + 6y$$
$$9x = 6y$$
$$3x = 2y$$
$$3\frac{dx}{dt} = 2\frac{dy}{dx}$$
$$3\cdot\frac{dx}{dt} = 2\cdot 5$$
$$\frac{dx}{dt} = \frac{10}{3}$$

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + 5 = \frac{10}{3} + 5 = \boxed{\frac{25}{3}\ \frac{ft}{sec}}$$

II



$$\frac{15}{6} = \frac{x+y}{y}$$

$$\frac{dx}{dt} = 5$$

find $\frac{d}{dt}(x+y)$

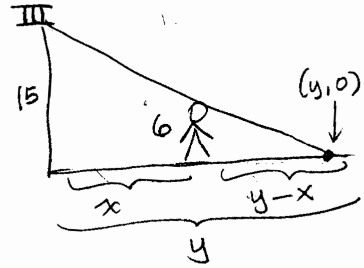
$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{dy}{dt}$$

cross mult

$$15y = 6x + 6y$$
$$9y = 6x$$
$$3y = 2x$$
$$3\frac{dy}{dt} = 2\frac{dx}{dt}$$
$$3\frac{dy}{dt} = 2\cdot 5$$
$$\frac{dy}{dt} = \frac{10}{3}$$

$$\frac{d}{dt}(x+y) = 5 + \frac{dy}{dt} = 5 + \frac{10}{3} = \boxed{\frac{25}{3}\ \frac{ft}{sec}}$$

III



$$\frac{15}{6} = \frac{y}{y-x}$$

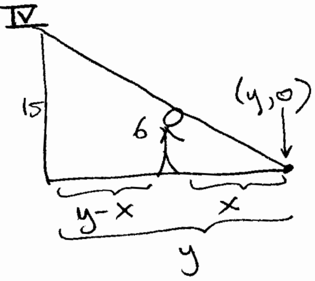
$$\frac{dx}{dt} = 5$$

find $\frac{dy}{dt}$

cross mult:

$$15y - 15x = 6y$$
$$9y = 15x$$
$$3y = 5x$$
$$3\frac{dy}{dt} = 5\cdot\frac{dx}{dt}$$
$$3\cdot\frac{dy}{dt} = 5\cdot 5$$
$$\boxed{\frac{dy}{dt} = \frac{25}{3}\ \frac{ft}{sec}}$$

IV



$$\frac{15}{6} = \frac{y}{x}$$

$$\frac{d}{dt}(y-x) = 5$$

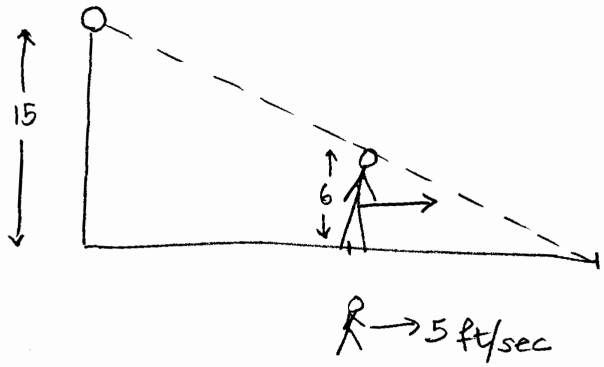
Find $\frac{dy}{dt}$

$$\frac{d}{dt}(y-x) = \frac{dy}{dt} - \frac{dx}{dt}$$
$$\frac{dy}{dt} - \frac{dx}{dt} = 5$$
$$\frac{dy}{dt} - 5 = \frac{dx}{dt}$$

cross-mult

$$15x = 6y$$
$$5x = 2y$$
$$5\frac{dx}{dt} = 2\cdot\frac{dy}{dt}$$
$$5\left(\frac{dy}{dt} - 5\right) = 2\cdot\frac{dy}{dt}$$
$$5\frac{dy}{dt} - 25 = 2\frac{dy}{dt}$$
$$3\frac{dy}{dt} = 25$$
$$\boxed{\frac{dy}{dt} = \frac{25}{3}\ \frac{ft}{sec}}$$

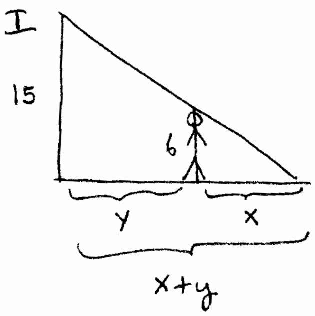
(9) cont



b) when he is 10 ft away, at what rate is the length of his shadow changing?

Here are Four ways to draw diagram → notice that rate of change of <u>shadow length</u> is a different quantity depending on set up.

I



$$\frac{15}{6} = \frac{x+y}{x}$$

$$\frac{dy}{dt} = 5$$

Find $\frac{dx}{dt}$

cross-mult:

$15x = 6x + 6y$

$9x = 6y$

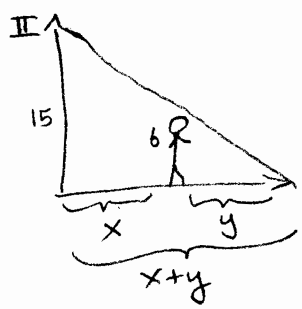
$3x = 2y$

$3 \cdot \frac{dx}{dt} = 2 \cdot \frac{dy}{dt}$

$3 \cdot \frac{dx}{dt} = 2 \cdot 5$

$$\boxed{\frac{dx}{dt} = \frac{10}{3} \text{ ft/sec}}$$

II



$$\frac{15}{6} = \frac{x+y}{y}$$

$$\frac{dx}{dt} = 5$$

Find $\frac{dy}{dt}$

cross-mult

$15y = 6x + 6y$

$9y = 6x$

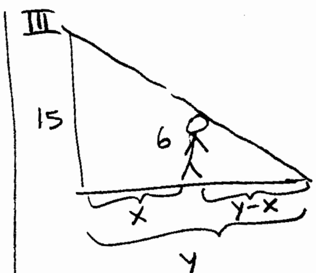
$3y = 2x$

$3\frac{dy}{dt} = 2 \cdot \frac{dx}{dt}$

$3 \cdot \frac{dy}{dt} = 2 \cdot 5$

$$\boxed{\frac{dy}{dt} = \frac{10}{3} \text{ ft/sec}}$$

III



$$\frac{15}{6} = \frac{y}{y-x}$$

$$\frac{dx}{dt} = 5$$

Find $\frac{d(y-x)}{dt}$

cross-mult

$15y - 15x = 6y$

$9y = 15x$

$3y = 5x$

$3\frac{dy}{dt} = 5\frac{dx}{dt}$

$3 \cdot \frac{dy}{dt} = 5 \cdot 5$

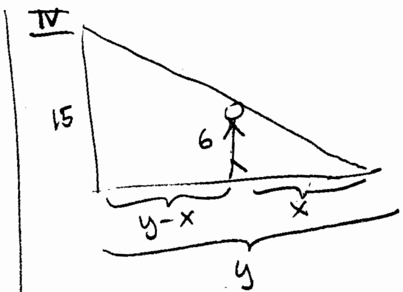
$\frac{dy}{dt} = \frac{25}{3}$

$\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt}$

$= \frac{25}{3} - 5$

$$\boxed{\frac{d(y-x)}{dt} = \frac{10}{3} \text{ ft/sec}}$$

IV



$$\frac{15}{6} = \frac{y}{x}$$

$$\frac{d}{dt}(y-x) = 5$$

Find $\frac{dx}{dt}$

$\frac{d}{dt}(y-x) = \frac{dy}{dt} - \frac{dx}{dt} = 5$

means $\frac{dy}{dt} = 5 + \frac{dx}{dt}$.

cross-mult

$15x = 6y$

$5x = 2y$

$5\frac{dx}{dt} = 2\frac{dy}{dt}$

$5\frac{dx}{dt} = 2\left(5 + \frac{dx}{dt}\right)$

$5\frac{dx}{dt} = 10 + 2\frac{dx}{dt}$

$3\frac{dx}{dt} = 10$

$$\boxed{\frac{dx}{dt} = \frac{10}{3} \text{ ft/sec}}$$

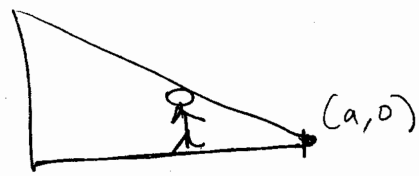
9 cont

Key points:

#1 Any set-up will work. You can change the set-up for part b)

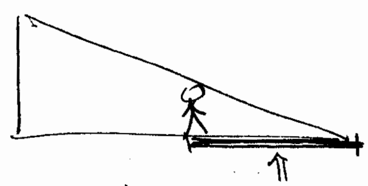
#2 Focus on what the question <u>asks</u>.

a) rate the tip of shadow moves



means $\frac{da}{dt}$, the rate of change of this coordinate.

b) rate of change of length of shadow



means $\frac{d(\text{shaded})}{dt}$.

#3 Some set-ups will make the desired quantity be: $\frac{d}{dt}(x-y)$

or $\frac{d}{dt}(x+y)$

or $\frac{d}{dt}(y-x)$

Remember the sum and difference rules for derivatives!

$$\frac{d}{dt}(x-y) = \frac{dx}{dt} - \frac{dy}{dt}$$

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$$

$$\frac{d}{dt}(y-x) = \frac{dy}{dt} - \frac{dx}{dt}$$

Related Rates

#1 Key: Every variable in this section can change as time changes, meaning, it's a function of time.

$x = x(t)$
$y = y(t)$
$r = r(t)$ etc.

#2 Key: When solving related rates problems, we are taking derivatives with respect to time, using chain rule.

$\frac{dx}{dt}$ = the rate x changes per unit of time.

$\frac{d\theta}{dt}$ = the rate angle θ changes per unit of time.

#3 Key: We will (often) be given numerical values for

rates $\frac{dx}{dt}$ (numerical value for rate means constant rate $\frac{dx}{dt}$.)

and/or variables x

while one quantity will be unknown – it will usually be a rate.

We need an equation connecting all these rates and variables so that we can substitute and solve for the one unknown.

Extras.

(A) Given $x^2 + y^2 = 25$

a) Find $\frac{dy}{dt}$ when $x=3$, $y=4$ $\frac{dx}{dt} = 8$.

Step 1: Differentiate WRT t.

$$2x \cdot \frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Step 2: Substitute known values.

$$2 \cdot 3 \cdot 8 + 2 \cdot 4 \cdot \frac{dy}{dt} = 0.$$

Step 3: Solve for unknown value.

$$\frac{dy}{dt} = \frac{-2 \cdot 3 \cdot 8}{2 \cdot 4} = \boxed{-6}$$

Related Rates

(B) Let A be the area of a circle of radius r that is changing with respect to time.
If $\frac{dr}{dt}$ is $3\,\frac{cm}{sec}$ and $r = 4$ cm what is $\frac{dA}{dt}$?

step 1: Geometry $\quad A = \pi r^2$

step 2 Diff WRT t $\quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

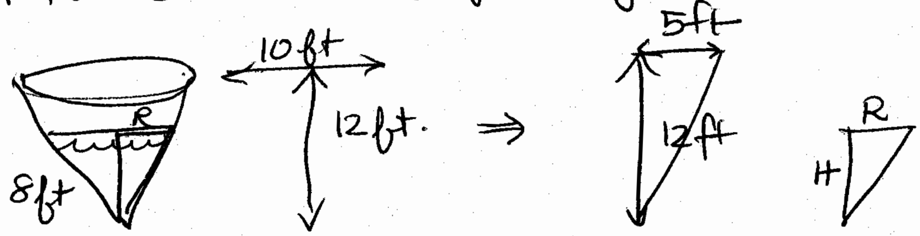
step 3 subst & solve

$$\frac{dA}{dt} = 2\pi \cdot 4 \cdot 3$$

$$\frac{dA}{dt} = \boxed{24\pi \ \frac{cm^2}{sec}}$$

(C) A conical tank (vertex down) is 10 feet across the top and 12 ft deep.
#22 Larson 9/e
If water is flowing into the tank at $10\ ft^3/min$, find the rate of change of the depth of the water when the water is 8 ft deep.

Step 1 Geometry:

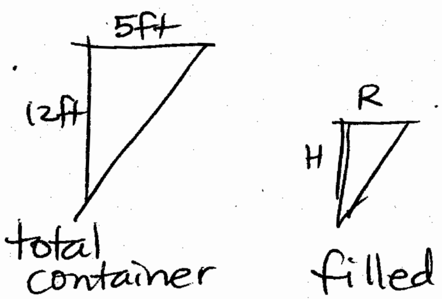


CONES ⟺ SIMILAR Δs

CAUTION: when problems involve filling or emptying containers, do NOT subst the size of the container. We want variable values for the size of the water.

The size of the container is used for the geometry, usually to eliminate a variable.

Similar Δs have proportional sides.



CAUTION: Use variables for moving parts until the very end of the work.

(c) cont $\frac{5}{12} = \frac{R}{H}$ means $5H = 12R$ $\Rightarrow$ $H = \frac{12}{5}R$

OR $R = \frac{5}{12}H$.

Do we want H or R in our geometry eqn?

Memorize volume of cone * → $\boxed{V = \frac{1}{3}\pi r^2 h}$

find rate of change of depth of water ← this is H.
So we want H, but no R.
Substitute $R = \frac{5}{12}H$.

$$V = \frac{1}{3}\pi\left(\frac{5}{12}H\right)^2 H$$

$$V = \frac{1}{3}\pi \cdot \frac{25}{144}H^3$$

$$V = \frac{25}{432}H^3$$

Step 2 Differentiate WRT time.

$$\frac{dV}{dt} = \frac{25}{144}\pi H^2 \frac{dH}{dt}.$$

Step 3: Subst: $10 = \frac{dV}{dt}$ and $8 = \cancel{H}$.

$$10\,\frac{ft^3}{min} = \frac{25}{144}\pi \cdot (8)^2 \cdot \frac{dH}{dt}.$$

Step 4 Solve for $\frac{dH}{dt}$:

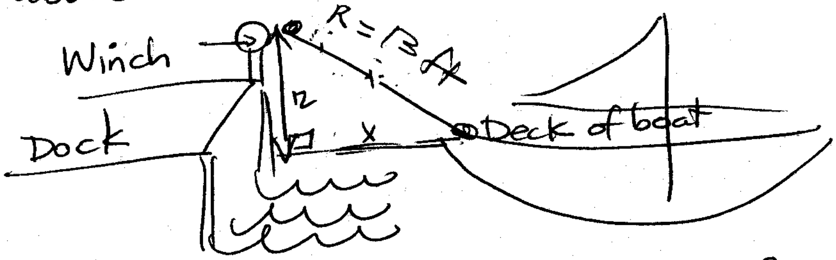
$$\frac{10 \cdot 144}{25 \cdot \pi \cdot 64} = \frac{dH}{dt}$$

$$\boxed{\frac{dH}{dt} = \frac{9}{10\pi}\ ft/min}$$ ≈ 2.83 ft/min. speed water is rising in the cone at ~~that~~ that moment

M250 2.6 Related Rates

(1) A boat is pulled into a dock using winch 12 ft above the deck of the boat.

#28 Larson ale



a) The winch pulls in rope at 4 ft/sec. Determine the speed of the boat when there are 13 ft of rope out.

Geometry $x^2+y^2=R^2$ $\quad$ y=12 never changes

$x^2+12^2=R^2$ $\quad$ x changes

$x^2+144=R^2$ $\quad$ R changes.

Diff $2x\frac{dx}{dt}+0=2R\frac{dR}{dt}$

$x\frac{dx}{dt}=R\frac{dR}{dt}$ $\qquad x^2+12^2=13^2$

$x=5$

Subst + Solve $5\cdot\frac{dx}{dt}=13\cdot(4)$

$\frac{dx}{dt}=\frac{13\cdot 4}{5}=\frac{52}{5}=\boxed{10.4\text{ ft/sec}}$

b) what happens to the speed of the boat as it gets closer to the dock?

$\cancel{2}(3.5)\frac{dx}{dt}=\cancel{2}\cdot(12.5)\cdot 4$ $\qquad x=\sqrt{12.5^2-12^2}=\sqrt{12.25}=3.5$

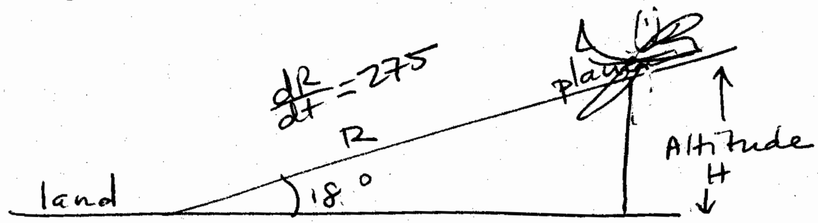
$\frac{dx}{dt}=\frac{(12.5)(4)}{(3.5)}=\frac{100}{7}\approx 14.29$ ft/sec

$\boxed{\text{It speeds up.}}$

M250 Related Rates

#47 Larson 9le E

An airplane is flying in still air with an airspeed of 275 mph. If it is climbing at an angle of 18°, find the rate at which it is gaining altitude.



Find $\frac{dH}{dt}$.

Geometry: $\sin\theta = \frac{H}{R}$

Subst $\theta = 18$: $\sin 18° = \frac{H}{R}$

$R \cdot \sin 18° = H$

Diff: $\sin 18° \cdot \frac{dR}{dt} = \frac{dH}{dt}$

Subst $\frac{dR}{dt}$ $(\sin 18°)(275) = \frac{dH}{dt}$

Calc & round $\frac{dH}{dt} = 84.97967$

$\boxed{84.9797 \text{ mph}}$